A simple construction of very high order non oscillatory compact schemes on unstructured meshes.

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Abstract

In [3] have been constructed very high order residual distribution schemes for scalar problems. They were using triangle unstructured meshes. However, the construction was quite involved and was not very flexible. Here, following [1], we develop a systematic way of constructing very high order non oscillatory schemes for such meshes. Applications to scalar and systems problems are given.

1 Introduction

We are interested in the approximation of the following model problem

$$\begin{array}{ll} \vec{\lambda}\cdot\nabla u = f & x\in\Omega \\ u=0 & x\in\Gamma^- \end{array} \tag{1}$$

where $\Omega\subset\mathbb{R}^d$ is a polygonal set, Γ^- is the inflow boundary

$$\Gamma^- = \{x\in\partial\Omega, \vec{\lambda}\cdot\vec{n}_x<0\}$$

and $\vec{n}_x$ is the local normal at the point $x\in\partial\Omega$.

We consider a conformal triangulation $\mathcal{T}_h$ which elements K are triangles, quads in 2D or tetrahedrons/hex in 3D. More general elements could, in principle, be considered. The parameter h denotes the maximum value of all the diameters h_K of the circumscribed circle/sphere to the elements K of $\mathcal{T}_h$. We also assume that the meshes are regular. The mesh $\mathcal{T}_h$ are assumed to be adapted to (1), i.e. Γ^- is a collection of edges/faces of $\mathcal{T}_h$. The space V_h^p is the set of continuous functions that, on each element K, are polynomials of degree p that vanishes on Γ^-.

The equation (1) is discretized by a variational formulation of the type : Let us give $p, q \in \mathbb{N}^\star$, find $u^h \in V_h^p$ such that for all $v^h \in V_h^q$,

$$a(u_h, v_h) = \ell(v_h). \tag{2}$$

The examples we are interested in are the SUPG schemes [6, 7] and the stabilized residual distribution schemes [1] which are in general non linear schemes, even for linear problems.

In the SUPG schemes, we take $p = q \in \mathbb{N}^\star$ and the relation (2) writes

$$\int_\Omega v^h \big(\vec{\lambda} \cdot \nabla u^h - f(x)\big) dx + h \int_\Omega \big(\lambda \cdot \nabla v^h\big) \, \big(\vec{\lambda} \cdot \nabla u^h - f(x)\big) dx = 0 \tag{3}$$

i.e.

$$\begin{aligned} a(v^h, u^h) &= \int_\Omega v^h \big(\vec{\lambda} \cdot \nabla u^h\big) dx + h \int_\Omega \big(\lambda \cdot \nabla v^h\big) \, \big(\vec{\lambda} \cdot \nabla u^h\big) dx \\ \ell(v^h) &= \int_\Omega v^h f(x) dx + h \int_\Omega \big(\lambda \cdot \nabla v^h\big) f(x) dx \end{aligned} \tag{4}$$

In the second example, the Residual Distribution schemes (RD schemes for short), we also take $q = p \in \mathbb{N}^\star$ but the formulation is completely different in order to account, for example, of a maximum principle. These schemes are described in section 2.

The solutions of (2) are obtained by an iterative scheme. The convergence of the iterative procedure is important for two reasons

1. The uniqueness of the solutions of (2) is essential to have a well posed problem, and one wishes to obtain a good approximation of the solution of (2).

2. One can show, and we recall this later in the text, that if the the problem (2) is not solved with enough precision, the formal accuracy of the scheme (2) is lost.

In this respect, the SUPG scheme is dissipative, and coercive in a proper norm, so that existence and uniqueness is guarantied. However, in the case of the RD schemes, this may be no longer true, at least for their unstabilized version. Indeed, a RD scheme can be constructed so that it is positivity preserving *but* in general, the solution of (2) may not be unique, see [1].

Coming back to the SUPG scheme, we see that the forms a and b are the sum of two terms, the forms

$$\begin{aligned} &a'(u^h, v^h) = \int_\Omega v^h \big(\vec{\lambda} \cdot \nabla u^h\big) dx \\ &\text{and} \\ &\ell'(v^h) = \int_\Omega v^h f dx \end{aligned} \tag{5}$$

that define the Galerkin formulation of (1). The problem (2) with a' and ℓ' is known to be very unstable. It is stabilized by adding a dissipative term q and a linear form b to keep the consistancy of the scheme,

$$
\begin{aligned}
&q(u^h,v^h)=h\sum_K \mathcal{D}_K(v^h,u^h), \quad \mathcal{D}_K(v^h,u^h)=\int_K \big(\vec{\lambda}\cdot\nabla v^h\big)\big(\vec{\lambda}\cdot\nabla u^h\big)dx \\
&b(v^h)=h\int_\Omega \big(\vec{\lambda}\cdot\nabla v^h\big)f(x)dx
\end{aligned}
\tag{6}
$$

The exact evaluation of $\mathcal{D}_K$ may be quite costly in practice. If in the case of $p=1$, the terms $\big(\vec{\lambda}\cdot\nabla u^h\big)\big(\vec{\lambda}\cdot\nabla v^h\big)$ can be evaluated with second order accuracy with only one point (the centroid). For $p=2$, the components of ∇v_h are of degree one, and an exact quadrature formula (for a constant velocity) $\vec{\lambda}$ is obtained with 3 quadrature points (the mid-points of the edges of the triangle) of a 5 point formula as indicated in Table 1. When $p=3$, $\left(\vec{\lambda}\cdot\nabla u^h\right)\left(\vec{\lambda}\cdot\nabla v^h\right)$ is of degree 4 and 7 quadrature points are needed. This can be seen from Table 1 where weights and quadrature points are displayed for triangular elements. For Q^k elements, the situation is worse.

Error $O(h^k)$	Weights	Λ_1	Λ_2	Λ_3
3	1/3	1/2	1/2	0
	1/3	0	1/2	1/2
	1/3	1/2	0	1/2
3	0.109951743655322	0.816847572980459	0.091576213509771	0.091576213509771
	0.109951743655322	0.091576213509771	0.091576213509771	0.816847572980459
	0.109951743655322	0.091576213509771	0.816847572980459	0.091576213509771
	0.223381589678011	0.108103018168070	0.445948490915965	0.445948490915965
	0.223381589678011	0.445948490915965	0.108103018168070	0.445948490915965
	0.223381589678011	0.445948490915965	0.108103018168070	0.445948490915965
5	0.225	1/3	1/3	1/3
	0.125939180544827	0.797426985353087	0.101286507323456	0.101286507323456
	0.125939180544827	0.101286507323456	0.101286507323456	0.797426985353087
	0.125939180544827	0.101286507323456	0.797426985353087	0.101286507323456
	0.13239415278850	0.470142064105115	0.470142064105115	0.059715871789770
	0.13239415278850	0.470142064105115	0.059715871789770	0.470142064105115
	0.13239415278850	0.470142064105115	0.059715871789770	0.470142064105115

Table 1: Examples of quadrature points and weights for triangles.

The question we are interested in this paper is the following. Given a scheme of the type (2),

$$a(u^h,v^h)=\ell(v^h)$$

what are the requirements about the forms q and b such that the scheme

$$a(u^h, v^h) + q(u^h, v^h) = \ell(v^h) + b(v^h) \tag{7}$$

is well posed, has provable error estimates in a well behaved norm ? How can q and b be chosen such that the evaluation of these terms is as simple as possible with the minimal number of operations ?

The schemes we are interested in, like (3) or the RD scheme, share several formal properties in common. Namely,

1. if u is a smooth solution of (1), then for any $v^h \in V_h^q$, we have

$$a(u, v^h) = \ell(v^h) \tag{8a}$$

   Moreover, if u^h denotes the solution of scheme, we have

$$a(u - u^h, v^h) = 0. \tag{8b}$$

   Note that this property, which is well known for the SUPG scheme, is also true for the RDS scheme, even if the RDS scheme is non linear.

2. From this, if u^h denotes now the interpolant of the exact solution u of (1), then the equivalent equation of the scheme is

$$a(u^h, v^h) - \ell(v^h) = O(h^{p+d}) \tag{9}$$

   from which we deduce the formal order of accuracy. Of course, in the case of the SUPG scheme, things can be made more rigorous.

These properties must remain intact.

In the first section, we explain in detail what is a RD scheme. The SUPG schemes are particular cases. The second section is devoted to the describe and discuss natural necessary conditions. The third section is devoted to examples and numerical results.

2 Examples of "unstabilised" schemes

The example of the Galerkin formulation of (1) is well known so we skip it. We give some details on the RD schemes that are less known.

We consider a conformal mesh, the generic element is denoted by K. The degrees of freedom are denoted by x_σ. In the case of a P^1 interpolant, they are just the vertices of the mesh. For a P^2 interpolant, we have to add the mid–edge points, etc. Obvious generalization can be described for other continuous elements such as the P^k or Q^k elements.

In order to construct a RD scheme for (1), on has first to construct "residuals" Φ_σ^K such that the two conditions are met :

1. Compact stenceil condition : $\Phi_\sigma^K(u^h) := \Phi_\sigma^K$ only depends on the values of u at the degrees of freedom in K,

2. Conservation condition : Φ_σ^K are such that

$$\sum_{\sigma\in K} \Phi_\sigma^K = \int_{\partial K} \vec{\lambda}\cdot\vec{n}\; u^h dx - \int_K f(x)dx := \Phi^K$$

   This is a conservation constraint.

The function u^h has to be solution of

$$\text{for any } \sigma, \qquad \sum_{K,\sigma\in K} \Phi_\sigma^K = 0. \tag{10}$$

As said previously, the SUPG schemes are examples of RD schemes, since they are exactly (10) with

$$\Phi_\sigma^K = \int_K \varphi_\sigma \big(\vec{\lambda}\cdot\nabla u^h - f\big)dx + h\int_K \big(\vec{\lambda}\cdot\nabla\varphi_\sigma\big)\big(\vec{\lambda}\cdot\nabla u^h - f\big)dx.$$

Of course these conditions (conservation and a compact stencil) are not enough to provide a working scheme in term of stability and accuracy. Here we foccus on the L^∞ stability and the residual property (8a)–(8b) which ensure formal accuracy. These two additional constraints are achieved by the following procedures[1].

One starts from a monotone scheme, say the Lax–Friedrichs one,

$$\Phi_\sigma^{K,LxF} = \frac{1}{N_K}\left(\Phi^K + \alpha_K \sum_{\sigma'\in K}(u_\sigma - u_{\sigma'})\right)$$

which is only first order, N_K represents the number of degree of freedom in K. Then we define

$$x_\sigma := \frac{\Phi_\sigma^K}{\Phi^K},$$

they sum up to unity thanks to the conservation relation and

$$\beta_\sigma^K := \frac{x_\sigma^+}{\sum\limits_{\sigma'\in K} x_{\sigma'}^+}. \tag{11}$$

There is no problem in the definition of β_σ^K since

$$\sum_{\sigma'\in K} x_{\sigma'}^+ \geq \sum_{\sigma'\in K} x_{\sigma'} = 1.$$

[1] Note that other RD scheme exist, they do not satisfy a L^∞ stability property. An example is the SUPG scheme, another one is the LDA scheme, see [5, 11].

The RD scheme is then defined by (10) with

$$\Phi_\sigma^K = \beta_\sigma^K \Phi^K. \tag{12}$$

The solution of (18)–(12) is sought for by an iterative method. The simplest one is

$$u_\sigma^{n+1} = u_\sigma^n - \omega_\sigma \sum_{K, \sigma\in K} \Phi_\sigma^K, \qquad \text{for all } \sigma \tag{13}$$

with $u_\sigma^0 = 0$ for example, and one hopes that $u_\sigma = \lim_{n\to+\infty} u_\sigma^n$. Thanks to the definition of β_σ^K, one can see that the sequence $\{u_\sigma^n\}_{n,\sigma}$ satisfy a maximum principle provided a CFL–like condition

$$0 \leq \omega_\sigma \max_{K,\sigma\in K} \left(|K| \max_{\sigma'\in K} \max_{x\in K} ||\nabla \varphi_{\sigma'}(x)|| \right) \leq 1.$$

Note that sharper estimates can be given, but this is not the point here.

The variational formulation of (10)–(12) is easily obtained. If one multiply (10) by $v(x_\sigma)$ and sum over all the degrees of freedom, one obtains

$$0 = \sum_K \Psi_K$$

with

$$\begin{aligned}\Psi_K &= \sum_{\sigma\in K} v_\sigma \beta_\sigma^T \int_K \big(\vec{\lambda}\cdot\nabla u^h - f\big)dx\\ &= \int_K v^h\big(\vec{\lambda}\cdot\nabla u^h - f\big)dx + \sum_{\sigma\in K} v_\sigma \int_K \big(\beta_\sigma^K - \varphi_\sigma\big)\big(\vec{\lambda}\cdot\nabla u^h - f\big)dx\end{aligned}$$

Since by definition

$$\sum_{\sigma\in T} \beta_\sigma^T = 1 = \sum_{\sigma\in T} \varphi_\sigma,$$

we have

$$\begin{aligned}\sum_{\sigma\in K} v_\sigma \int_K \big(\beta_\sigma^K - \varphi_\sigma\big)\big(\vec{\lambda}\cdot\nabla u^h - f\big)dx &= \frac{1}{(N_K-1)!}\sum_{\sigma,\sigma'} (v_\sigma - v_{\sigma'}) \int_K \Big(\gamma_{\sigma,\sigma'} - \psi_{\sigma,\sigma'}\Big)\big(\vec{\lambda}\cdot\nabla u^h - f\big)dx\\ &= \frac{h}{(N_K-1)!}\sum_{\sigma,\sigma'} \theta_{\sigma\sigma'} \frac{v_\sigma - v_{\sigma'}}{||x_\sigma \vec{x}_{\sigma'}||} \int_K \Big(\gamma_{\sigma,\sigma'} - \psi_{\sigma,\sigma'}\Big)\big(\vec{\lambda}\cdot\nabla u^h - f\big)dx\end{aligned}$$

with $\theta_{\sigma\sigma'} = ||x_\sigma \vec{x}_{\sigma'}||/h$ which is bounded since the mesh is regular, $\gamma_{\sigma,\sigma'} = \beta_\sigma^T - \beta_{\sigma'}^T$ and $\psi_{\sigma,\sigma'} = \varphi_\sigma - \varphi_{\sigma'}$.

The form a is

$$\begin{aligned}a(u^h, v^h) &= \int_K v^h\big(\vec{\lambda}\cdot\nabla u^h\big)dx\\ &\quad + \frac{h}{(N_K-1)!}\sum_{\sigma,\sigma'} \theta_{\sigma\sigma'} \frac{v_\sigma - v_{\sigma'}}{||x_\sigma \vec{x}_{\sigma'}||} \int_K \Big(\gamma_{\sigma,\sigma'} - \psi_{\sigma,\sigma'}\Big)\big(\vec{\lambda}\cdot\nabla u^h\big)dx\end{aligned} \tag{14}$$

and ℓ is

$$
\begin{aligned}
\ell(, v^h) = &\int_K v^h f dx \\
&+ \frac{h}{(N_K-1)!} \sum_{\sigma,\sigma'} \theta_{\sigma\sigma'} \frac{v_\sigma - v_{\sigma'}}{||x_\sigma \vec{x}_{\sigma'}||} \int_K \Big(\gamma_{\sigma,\sigma'} - \psi_{\sigma,\sigma'} \Big) f dx
\end{aligned}
\tag{15}
$$

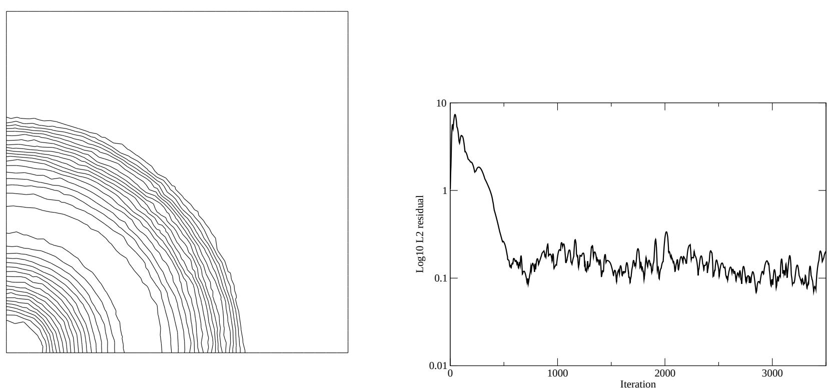
which have the same structure as (4).

The problem of this scheme is that even though the iteration (13) is L^∞ stable, it does not converge in general. The same conclusion holds for more involved iterative scheme and the reason is that (10)–(12) is not well posed except for very special situations.

An example is given for a second order (hence P^1 interpolation) using the local Lax Friedrichs scheme (12) on

$$
\begin{aligned}
&-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0 && (x,y) \in [0,1]^2 \\
&u(x,0) = \begin{cases} -\sin\left(\pi \dfrac{x-0.7}{0.6}\right) & \text{if } x \in [0.1, 0.7] \\ 0 & \text{else} \end{cases}
\end{aligned}
\tag{16}
$$

The convence history and a solution is given on Figure 1 The solution of Figure 1



Solution   Convergence history

Figure 1: Solution of (1) with the scheme (12) with β defined by (11).

is obviously not a second order accurate approximation of (16). The next section is devoted to describe a simple modification of the scheme. This problem has already been solved in [1] for second order schemes, we show how to extend the method in a simple and efficient way.

3 Construction and discussion

Let us consider the problem

$$a(u^h, v^h) = \ell(v^h) \tag{17a}$$

with a and ℓ given by

$$\begin{aligned} a(u^h, v^h) &= \sum_{K\in\mathcal{T}_h} a_K(u^h, v^h) \\ a_K(u^h, v^h) &= \int_K v_h\big(\vec{\lambda}\cdot\nabla u^h\big)dx + hb_K(u_h, v_h) \\ \ell(v^h) &= \sum_{K\in\mathcal{T}_h} \ell_K(v^h) \\ \ell_K(v^h) &= \int_K v_h f dx + hl_K(v_h) \end{aligned} \tag{17b}$$

We assume that a, ℓ, a_K and ℓ_K satisfy the following assumptions :

Assumption 3.1. *1. a_K and l_K are linear in v^h.*

2. if u is the solution of (1),

$$a(u, v^h) = \ell(v^h)$$

for any $v^h \in V_h$ and

$$a(u - u^h, v^h) = 0$$

for any $v^h \in V_h$. More precisely, because of the structure of the forms a and b, we assume that for any K, and any $v_h \in V_h^p$,

$$a_K(u, v^h) = l_K(v_h)$$

and

$$h\ a_K(u - u^h, v_h) = O(h^{p+d}).$$

These assumtions are true for the SUPG and RD schemes. Moreover, for these two schemes, we have the conservation constraint

$$a_K(u^h, 1) = l_K(1) = 0$$

for any K.

Remark 3.1 (About the linearity assumption)**.** *The problem* (1) *is linear. All what is said here can be extended to the non linear case, and the linearity assumption still holds.*

The scheme writes in the RD form (14). To see this, we consider the list of degrees of freedom $\{x_\sigma\}$. For piecewise linear interpolant and triangular elements or Q^1 interpolant, they are just the vertices of the mesh. For quadratic interpolant and triangular meshes, they are the vertices of the mesh and the mid–edges points, etc. The Lagrange interpolant of degree p associated to a given degree of freedom x_σ is denoted as φ_σ^p. We have

1. $\varphi^p_\sigma(x_{\sigma'}) = \delta_\sigma^{\sigma'}$,
2. φ^p_σ is continuous,
3. for any element T, the restriction of φ_σ to T is a polynomial of degree d.

By definition, any $u^h \in V_h^p$ can be written as

$$u^h = \sum_\sigma u^h(x_\sigma)\varphi_\sigma^p,$$

and $v^h \in V_h^q$ can be written as

$$v^h = \sum_\sigma v^h(x_\sigma)\varphi_\sigma^q,$$

so that the scheme can be reformulated as finding $u^h \in V_h^p$ such that for any σ, we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\Bigg\{\int_K \varphi^p_\sigma\big(\vec{\lambda}\cdot\nabla u^h\big)dx + h_K a_K(\varphi_\sigma, u^h)\\ + \sum_{T\subset\mathcal{T}_h}\int_T f(x)\varphi^q_\sigma(x)dx + h_K l_K(\varphi_q)\Bigg\} = 0\end{aligned} \tag{18}$$

Such a scheme is rewritten as a RD scheme with the residual

$$\Phi_\sigma^T = \int_K \varphi^p_\sigma\big(\vec{\lambda}\cdot\nabla u^h - f\big)dx + h_K\Big(a_K(\varphi_\sigma, u^h) - h_K l_K(\varphi_\sigma)\Big) \tag{19}$$

the conservation constraint is automatically satisfied because $\sum_{\sigma\in T}\varphi^p_\sigma = 1$ and using the linearity with respect to v_h,

$$\begin{aligned}\sum_{\sigma\in K}\Phi_\sigma^K &= \int_K \big(\vec{\lambda}\cdot\nabla u^h - f\big)dx + h_K\left(a_K\big(u_h, \sum_{\sigma\in K}\varphi_\sigma\big) - l_K\big(\sum_{\sigma\in K}\varphi_\sigma\big)\right)\\ &= \int_K \big(\vec{\lambda}\cdot\nabla u^h - f\big)dx\\ &= \Phi_K\end{aligned}$$

Formally, the scheme is accurate because if that if $w^h \in V_h^p$ is the interpolant of the exact solution of (1) assumed to be smooth, then $\Phi_\sigma^T(w^h) = O(h^{p+d})$. In fact, for RD scheme, this property is true because $\Phi^T(w^h) = O(h^{p+d})$ since

$$\begin{aligned}\Phi^T(w^h) &= \int_{\partial T}\vec{\lambda}\cdot\vec{n}w^h dx - \int_K f dx\\ &= \int_{\partial T}\vec{\lambda}\cdot\vec{n}\big(w^h(x) - u(x)\big)dx\\ &= O(h^{d-1})\times O(h^{p+1}) = O(h^{p+d}).\end{aligned}$$

Using this remark, the relation (9) follows for regular meshes.

A first scheme of the type (7) can be obtained by perturbing the residual (19) as

$$\left(\Phi_\sigma^T\right)^\star = \Phi_\sigma^T + \psi_\sigma^T \tag{20}$$

with the constraint $\sum\limits_{\sigma\in T} \psi_\sigma^T = 0$ to ensure conservation. The formal accuracy property is also conserved if

$$\psi_\sigma^T(w^h) - f = O(h^{p+d}) \tag{21}$$

whenever $\vec{\lambda}\cdot\nabla w^h = O(h^p)$.

A first example is obviously given by

$$\psi_\sigma^T = \theta h \int_K \big(\vec{\lambda}\cdot\nabla\varphi_\sigma^q\big)\big(\vec{\lambda}\cdot\nabla u_h - f\big)dx \tag{22}$$

where θ is chosen such that

$$\int_\Omega \left(\vec{\xi}\cdot\nabla u\right)\left(\vec{\lambda}\cdot\nabla u\right)d\mathbf{x} + \theta\int_\Omega \left(\vec{\lambda}\cdot\nabla u\right)^2 d\mathbf{x} \geq 0. \tag{23}$$

Under this condition, the iterative scheme (13) is convergent when $n \to +\infty$.

In (22), ψ_σ^T is evaluated by a quadrature formula of exact order,

$$\int_K \big(\vec{\lambda}\cdot\nabla\varphi_\sigma^q\big)\big(\vec{\lambda}\cdot\nabla u_h\big)dx = \sum_{x_{\text{quad}}} \omega_{\text{quad}}\big(\vec{\lambda}\cdot\nabla\varphi_\sigma^q(x_{\text{quad}})\big)\big(\vec{\lambda}\cdot\nabla u_h(x_{\text{quad}}) - f(x_{\text{quad}})\big)\big) \tag{24}$$

The question is now : given a formula of the type

$$\psi_\sigma^T = \theta_T\; h_K|T|\left(\sum_{x_{\text{quad}}} \omega_{\text{quad}}\big(\vec{\lambda}\cdot\nabla\varphi_\sigma^q(x_{\text{quad}})\big)\big(\vec{\lambda}\cdot\nabla u_h(x_{\text{quad}} - f(x_{\text{quad}}))\big)\right), \tag{25}$$

what are the requirements on the points x_{quad} and the weights ω_{quad}, so that we still have the inequality (23) and the accuracy condition (21) ?

Accuracy constraint. Assuming that the solution of (1) is smooth enough, we have on T

$$w^h - u = O(h^{p+1}) \qquad \text{and } \nabla\big(w^h - u\big) = O(h^p),$$

and for a regular mesh

$$\nabla\varphi_\sigma = O(h^{-1})$$

so that for any x_{quad},

$$\big(\vec{\lambda}\cdot\nabla\varphi_\sigma^q(x_{\text{quad}})\big)\big(\vec{\lambda}\cdot\nabla w_h(x_{\text{quad}} - f(x_{\text{quad}})\big) = O(h^{p-1})$$

so that

$$\psi_\sigma^T = h \times O(h^d) \times O(h^{p-1}) = O(h^{p+d}).$$

The formal accuracy is automatically guarantied.

Constraints on the weights and the points $x_{\mathbf{quad}}$. In order to have the inequality (23), a necessary condition is that the quadratic form

$$q_K(v_h) := \sum_{x_{\text{quad}}} \omega_{\text{quad}} \big(\vec{\lambda} \cdot \nabla v_h(x_{\text{quad}})\big)^2$$

must be positive definite whenever the polynomial $\vec{\lambda} \cdot \nabla v^h \neq 0$.

A sufficient condition is

$$\begin{aligned} &\text{for all } x_{\text{quad}}, \omega_{\text{quad}} > 0 \\ &\vec{\lambda} \cdot \nabla v_h(x_{\text{quad}}) = 0 \text{ for each } x_{\text{quad}}, \text{ then } \vec{\lambda} \cdot \nabla v_h = 0. \end{aligned} \tag{26}$$

Under these conditions, there exist constants $C_{1,q}$ and $C_{2,q}$ such that

$$C_{1,q} q_K(v_h) \leq h_K \int_K \big(\vec{\lambda} \cdot \nabla v_h\big)^2 dx \leq C_{2,q} q_K(v_h) \tag{27}$$

because P^q is a finite dimensional space, hence

$$Q(v_h) = \sum_K q_K(v_h)$$

defines a norm on V_h which is equivalent to the norm $v_h \mapsto \int_K \big(\vec{\lambda} \cdot \nabla v_h\big)^2 dx$.

We have shown the following result

Proposition 3.2. *If a and ℓ are defined by* (17b)*, under the assumptions 3.1 and provided that the conditions* (26) *hold, for each element K, there exists $\theta_{K,0} > 0$ such that the scheme* (17a) *for $\theta_K > \theta_{K,0}$. is well posed and .*

Proof. The scheme writes in variational formulation : find $u_h \in V_h$ such that for all $v_h \in V_h$,

$$a'(u_h, v_h) = \ell(v_h)$$

with

$$a' = a + b, \qquad \ell' = \ell$$

where

$$b(u^h, v^h) = \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in K} v_\sigma \psi_\sigma^K$$

with ψ_σ^K defined by (25) and Using the scalar product

$$\langle u_h, v_h \rangle = \sum_{K \in \mathcal{T}_h} |K| \bigg(\sum_{\sigma \in K} u_\sigma v_\sigma \bigg),$$

The iterative scheme (13) writes,

$$u^{n+1} = u^n - \omega\big(Au^n - F\big)$$

with

$$\langle Au^h, v^h\rangle := a(u^h, v^h), \qquad \langle F, v^h\rangle = \ell(v_h).$$

The scheme is convergent if

$$||Id - \omega A|| < 1.$$

A necessary condition is that for any v^h,

$$-2\langle Av^h, v^h\rangle + \omega ||Av^h|| \leq 0$$

for some $\omega > 0$. This condition needs

$$a(v^h, v^h) = \langle Av^h, v^h\rangle > 0 \tag{28}$$

for any v^h. Coming back to the problem,

$$a(v^h, v^h) = \int_{\partial\Omega^+} \vec{\lambda}\cdot\vec{n} v_h^2 dl + \sum_K h_K \bigg(a_K(v_h, v_h) + \theta_H q_K(v_h)\bigg),$$

hence a necessary condition for haveing (28) is that on any K we have

$$a_K(v_h, v_h) + \theta_H q_K(v_h) > 0$$

From assumption 3.1, we see that

$$\text{Ker } q_K = \{v_h \in P_K, q_K(v_h) = 0\} \subset \text{Ker } a_K = \{v_h \in P_K, a_K(v_h) = 0\},$$

so that which means that, since q_K is positive definite, the scheme is contractant provided that

$$\theta_H > \theta_{K,0} = \min\left(0, -\frac{\sup\limits_{v_h \in P^q} a(v^h, v^h)}{\inf\limits_{v_h \in P^q, v_h \notin \text{Ker} q_K} q_K(v^h)}\right) \in \mathbb{R}$$

for each K. □

4 Examples and numerical illustrations

4.1 Accuracy study

We apply the method on a simple linear problem, namely

$$\begin{aligned} &\frac{\partial u}{\partial y} = 0 && (x, y) \in [0, 1]^2 \\ &u(x, y) = \sin(\pi x)^2 && x = 0 \end{aligned} \tag{29}$$

for which the solution is simply $u(x, y) = \sin(\pi\sqrt{x^2 + y^2})^2$.

We have run the formaly second order scheme, third order and fourth order schemes with respectively 1, 3, 6 "quadrature points" in (25).

For the third order scheme, the "quadrature" points are simply the vertices of the triangle. For the third order case, we have chosen the vertices and the mid point edges. The weights are 1, 1/3 and 1/6 respectively. The results, in term of accuracy, are independant of choices of the "quadrature" points, provided that (24) defines a positive definite quadrature form. The constant θ in (25) is set to unity. The results are displayed in table 2. We see that the expected

$h = 1/N$	L^2	rate	L^∞	rate
25	$0.50493\,10^{-2}$	–	$0.30340\,10^{-1}$	–
50	$0.14684\,10^{-2}$	1.78	$0.12726\,10^{-1}$	1.25
75	$0.74684\,10^{-3}$	1.66	$0.82311\,10^{-2}$	1.07
100	$0.41019\,10^{-3}$	2.08	$0.52882\,10^{-2}$	1.54
Second order accurate results				
25	$0.32612\,10^{-4}$	–	$0.15748\,10^{-3}$	–
50	$0.48741\,10^{-5}$	2.742	$0.31276\,10^{-4}$	2.33
75	$0.13334\,10^{-5}$	3.19	$0.11363\,10^{-4}$	2.49
100	$0.66019\,10^{-6}$	2.44	$0.46897\,10^{-5}$	3.07
Third order accurate results				
25	$0.20860\,10^{-5}$	–	$0.12811\,10^{-4}$	–
50	$0.17001\,10^{-6}$	3.61	$0.17880\,10^{-5}$	2.84
75	$0.27027\,10^{-7}$	4.53	$0.26772\,10^{-6}$	4.68
100	$0.91462\,10^{-8}$	3.76	$0.95526\,10^{-7}$	3.58
Fourth order accurate results				

Table 2: Accurary results for (29). The third order accurate scheme uses the vertices in (25). The fourth order scheme uses the vertices and the mid–points (25). In each case, $\theta = 1$.

order of accuracy is met in each case. If now we repeat the same experiment with a smaller number of "quadrature" points, the accuracy is degraded and the

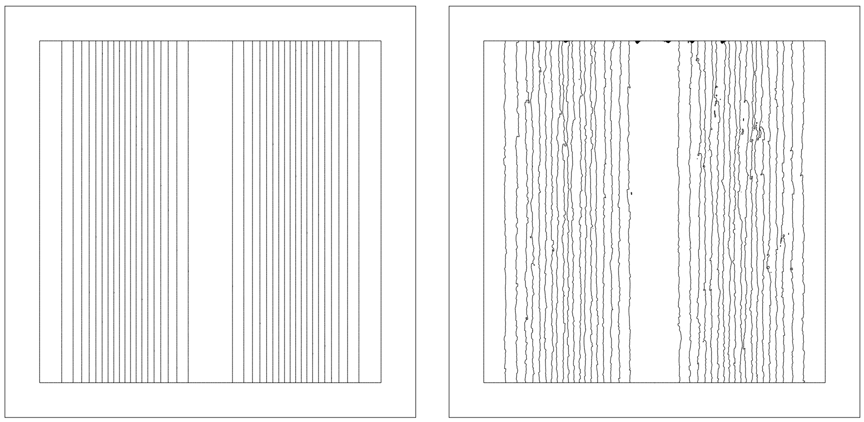
$h = 1/N$	L^2	rate	L^∞	rate
25	$0.25122\,10^{-1}$	–	0.42887	–
50	$0.12935\,10^{-1}$	0.9577	0.39237	0.1283
100	$0.83978\,10^{-2}$	0.6232	0.43656	−0.1540

Table 3: Accurary results for (29). The "third" order accurate scheme uses the gravity center in (25). In each case, $\theta = 1$.

results are only first order accurate or the scheme is only consistant, see tables 3 and 4. This can also be seen visually on Figure 2.

$h = 1/N$	L^2	rate	L^∞	rate
25	$2.17274\,10^{-2}$	–	0.10644	–
50	$1.13486\,10^{-2}$	0.8989	$7.94628\,10^{-2}$	0.9370
100	$5.83347\,10^{-3}$	0.9595	$4.16117\,10^{-2}$	0.9601

Table 4: Accurary results for (29). The "fourth" order accurate scheme uses the vertices in (25). In each case, $\theta = 1$.



3 points 1 point

Figure 2: Isolines of the solution of (29) when 1 point or 3 points are used in (25). The baseline scheme is formaly third order. All the degrees of freedom are represented.

4.2 The non linear case

The second example is the solution of the problem

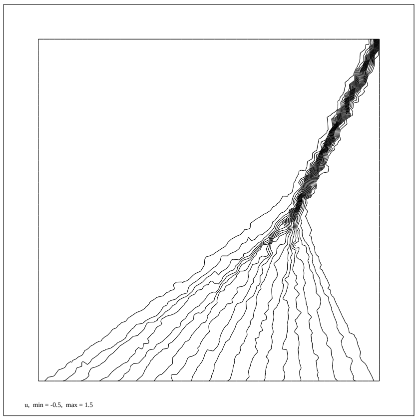
$$\begin{aligned} \tfrac{1}{2}\frac{\partial u^2}{\partial x} + \frac{\partial u}{\partial y} = 0 & \qquad \text{if } x \in [0,1]^2 \\ u(x,y) = 1.5x - 0.5 & \qquad \text{when } y = 0 \text{ or } x \in \{0,1\} \end{aligned} \tag{30}$$

The solution consists in a compression merging into a shock which foot is located at $(0.5, 0.75)$. Several schemes are tested. We only represent the solutions obtained by the formaly third order scheme since the behavior for the fourth order one is the same. The "quadrature" points are again the vertices of the elements with and without the centroid depending on if we take 3 or 4 points.

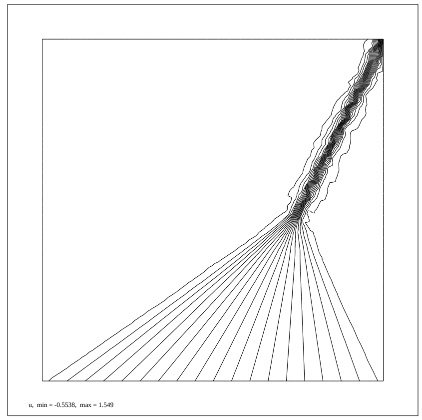
On Figure 3, we represent the isolines of the scheme when θ is set to 0, 1 or

$$\theta = \frac{|T|}{\sum\limits_{vertices} |k_i|} \min\left(1, \frac{\sqrt{|T|}\left(\sum\limits_{vertices} |k_i|\right)}{\left|\int_{\partial T} \left(n_x \frac{u^2}{2} + n_y u\right) dl\right| + \epsilon}\right). \tag{31}$$

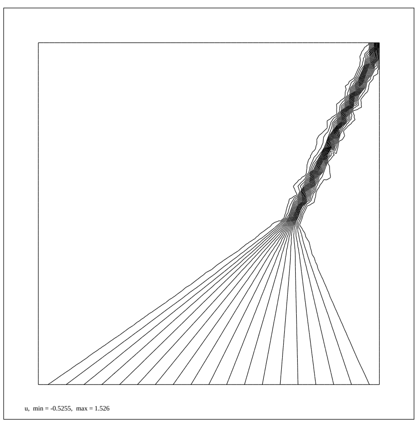
In (31), if n_x^i and n_y^i are the components of the inward normal opposite the vertex i in the triangle, $k_i = n_x^i \frac{u^2}{2} + n_y^i u$ and $\epsilon = 10^{-10}$. Once again, the same conclusions hold : 3 points are necessary to get accurate results. We compare the solutions depending on which option is chose (3/4 quad points, the choice of θ). To do this, we make cross–section at $y = 0.25$, i.e. in the fan, and $y = 0.75$, i.e. in the discontinuity.



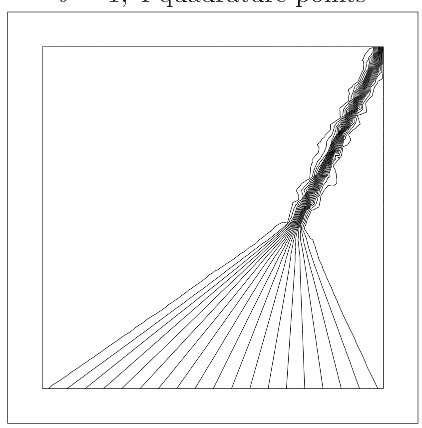
$\theta = 0$



$\theta = 1$, 4 quadrature points



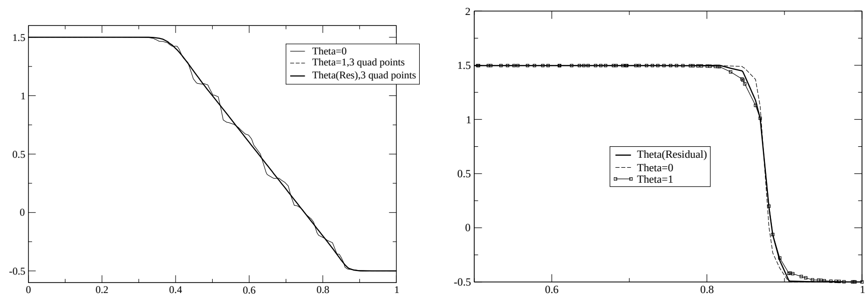
θ defined by (31), 4 quad points



θ defined by (31), 3 quad points

Figure 3: Results obtained for problem (30) with various choices of θ and quadrature points. In the case of 4 quadrature points we have chosen the centroid (weight $-27/48$), and the points of coordinates $(0.6, 0.2, 0.2)$, (0.2,0.6,0.2), $(0.2, 0.2, 0.6)$ with weights 25/48. In the case of 3 quadrature points, they are simply the vertices of the triangle with the weights 1/3.

The results of Figure 4 show that if $\theta = 0$, the oscillations visible in Figure 3 are not a manifestation of an instability, the scheme is overcompressive. When $\theta = 1$ or is chosen as (31), there is no difference in the solution, whatever the number of quadrature points. In Figure 4, we plot the result at $y = 0.75$. If we



cut at $y = 0.25$ cut at $y = 0.75$

Figure 4: Cross–section of the solution at $y = 0.25$ and $y = 0.75$. For $y = 0.25$, the solution corresponding to 3 and 4 quadrature points and $\theta \neq 0$ are undistinguishable. The curve labelled $\theta(Residu)$ corresponds to the choice (31). Some difference appears for the cross section at $y = 0.75$. The choice (31) appears to be a good compromise.

add the addition of the term (25), the scheme is no longer formaly monotonicity preserving, but the Figure 4 indicate that no undershoot nor overshoot are created. The same figure also indicate that the choice (31) is the best compromise between accuracy and stability. The effect of this term is that when the solution is smooth, $\theta \simeq 1$ while $\theta \simeq 0$ in the discontinuity.

4.3 Euler equations

The last examples that we show are for the Euler equations. Details about the scheme can be found in [2] in particular about the way equation (12) is implemented. The method has been implemented only for P_2 element so far again with 3 "quadrature" points. The first example is a supersonic jet with $M = 2.4$ on the bottom and $M = 4.4$ on the top. The solution, see Figure 5, is made of a shock wave followed by a contact and a fan. On Figure 6, we show the effect of adding and removing the term (25). We can also see the increase of accuracy. On Figure (7), we have run the first order, second order and third order RD schemes with the same number of degrees of freedom, namely the vertices and the mid–points of the mesh. A last example is a 4 state shock tube problem (configuration 12 of [8]). This case is time dependant, but we can compute the solution at time t since the solution is self–similar, $U(x, y, t) = V(\frac{x}{t}, \frac{y}{t})$. The function $V(\xi, \nu)$ satisfies

$$-\xi V_\xi - \nu V_\nu + \text{div}_{(\xi,\nu)} F(V) = 0.$$

The case has been chosen that the boundary condition can easily be computed analyticaly. The scheme is the same as before, but we modify the definition of

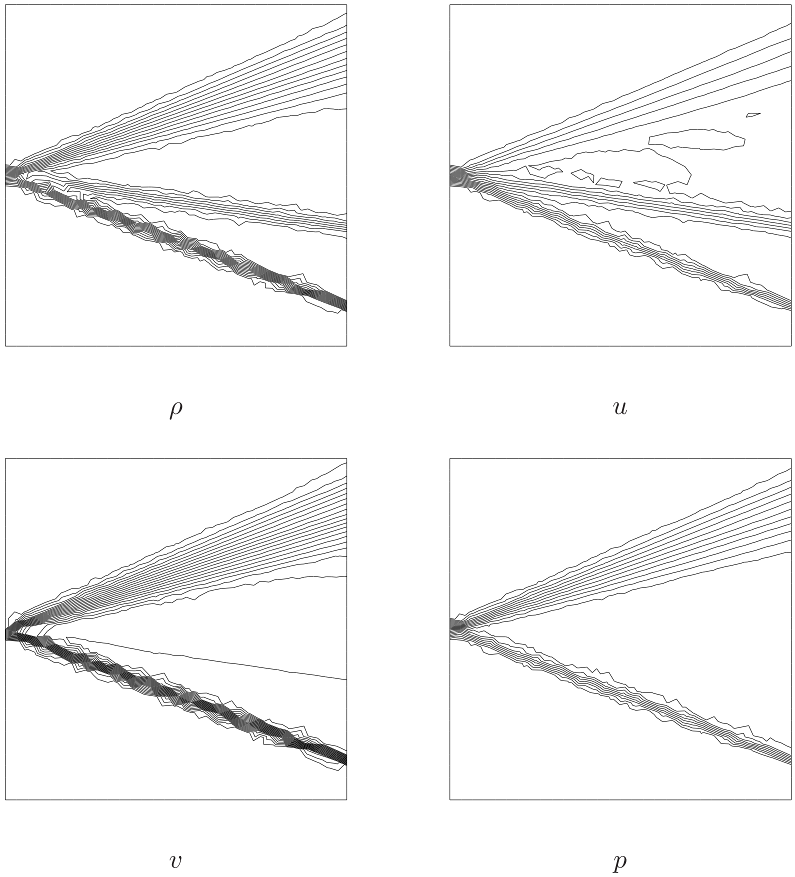


Figure 5: Supersonic ject, third order solution

the total residual by

$$\Phi^T := \int_T \Big(-\xi V_\xi - \nu V_\nu + \mathrm{div}_{(\xi,\nu)} F(V) \Big) d\xi d\nu.$$

This integral is evaluated by

$$\Phi^T = \int_{\partial T} \Big(F(V) \cdot \vec{n} - (\xi,\nu) \cdot \vec{n} \Big) dl + \int_T V(\xi,\nu) d\xi d\nu.$$

Again we see the improvement obtained by adding the term (25). The scheme is very robust and non oscillatory, despites the interaction between many waves.

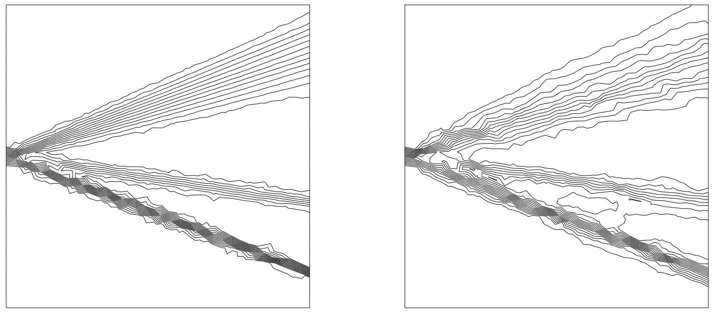


Figure 6: With and without dissipation, density isolines.

5 Conclusion

In this paper, we have discussed a simple way to construct simple and accurate very high order residual distribution schemes. A theoretical discussion is provided which is confirmed by numerical experiments on scalar problems and the Euler equations. We have foccused on schemes like Residual Distribution schemes, bu we believe however that the method we present in this paper can be adapted to other type of schemes. Note also that it shares common features with the work of Corre and Lerat, see [9, 10, 4] for example.

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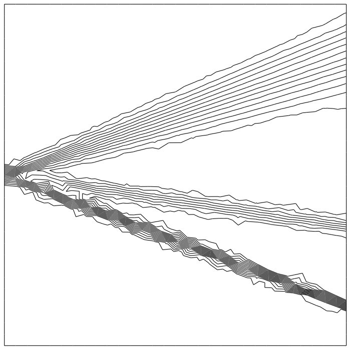
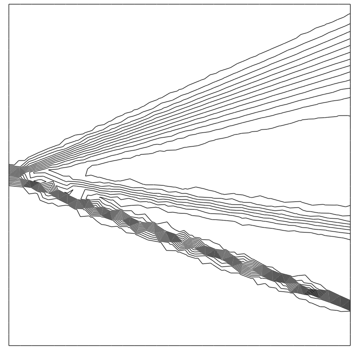
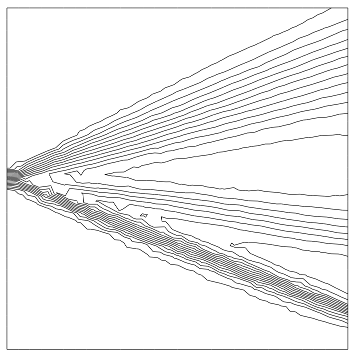




O3 O2



O1

Figure 7: Comparison between 1st, second and third order, same degrees of freedom

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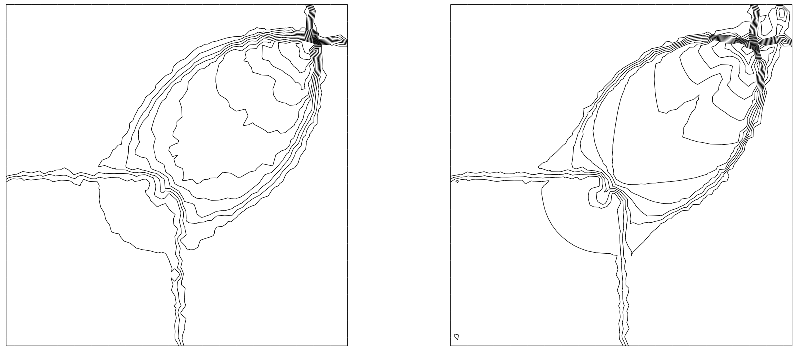

Figure 8: 4-state Riemann problem, comparison stabilized, unstabilized solution on a coars mesh

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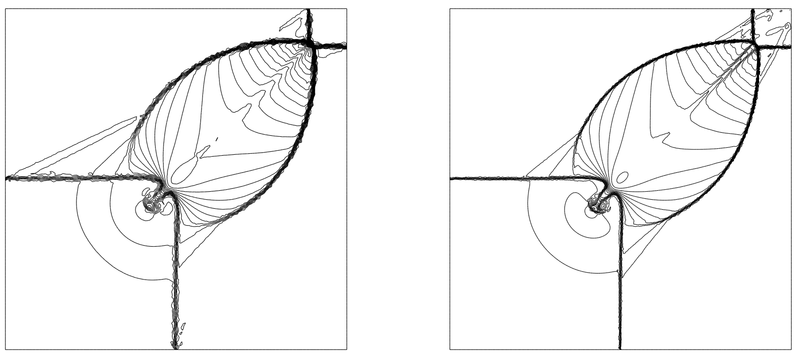

101 × 101   201 × 201

Figure 9: Convergence study 101×101 and 201×201, density isolines.